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**ANALYSIS TECHNIQUES FOR RESIDUAL  
ACCELERATION DATA**

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16. Abstract  Various aspects of residual acceleration data are of interest to low-gravity experimenters. Maximum and mean values and various other statistics can be obtained from data as collected in the time domain. Additional information may be obtained through manipulation of the data. Fourier analysis is discussed as a means of obtaining information about dominant frequency components of a given data window. Transformation of data into different coordinate axes is useful in the analysis of experiments with different orientations and can be achieved by the use of a transformation matrix. Application of such analysis techniques to residual acceleration data provides additional information than what is provided in a time history and increases the effectiveness of post-flight analysis of low-gravity experiments.					
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## TECHNICAL MEMORANDUM

### ANALYSIS TECHNIQUES FOR RESIDUAL ACCELERATION DATA

#### I. INTRODUCTION

There are various aspects of observational data that may be of interest to an investigator, e.g., mean, variance, and minimum and maximum values. Observational data such as biomedical data, economic data, seismic data, and accelerometer data are recorded as either continuous time functions or discrete time series. While statistics such as those mentioned above can be obtained from data in this form, additional information can often be obtained by looking at the data from a different perspective, such as can be obtained by transformation of data into a different domain or into different coordinate axes. Of particular interest to us is the analysis of residual acceleration data collected in orbiting space laboratories. A thorough understanding of such data and the ability to manipulate the data will allow the characterization of orbiters so that investigators can better understand the results of low-gravity experiments.

Most time functions can be considered as the sum of sinusoidal and co-sinusoidal terms of various frequencies (all harmonics of some fundamental frequency) and a steady state term. Spectral or Fourier analysis consists of a transformation of data in the time domain into the frequency domain in which the relative strengths of the frequency components of the data can be studied. Although the equations used to perform such transformations were originally applied to continuous, periodic functions, the theory can be adapted to the analysis of discrete, aperiodic time series.

The orientation of recording axes can be an important factor in the collection of observational data. It may be of interest to an investigator to look at data in terms of a set of axes other than that in which the data were collected. Such analysis is performed by means of a transformation of coordinate axes in which data in a given system are referred to in terms of some new set of coordinate axes. A transformation may be useful in the analysis of multiple experiments conducted in different orientations.

Before any such manipulation can be performed on data, the data must be collected. Various details associated with the sampling interval used in the acquisition of data and the window lengths used in the analysis of data are discussed in the next section. The adaptation of Fourier theory to the analysis of discrete, aperiodic time series is discussed in the Spectral Analysis section. Also discussed in that section are the power spectral density function and the application of spectral analysis to residual acceleration data. The transformation of data from one set of coordinate axes to another by means of a transformation matrix and how this can be useful in the analysis of low-gravity data are discussed in the Transformation of Coordinate Axes section. A symbol nomenclature is provided in Appendix A, and detailed derivations are included in Appendix B.

## II. SAMPLING CONSIDERATIONS AND FREQUENCY LIMITS

In the analysis of observational data, certain restrictions are imposed by the length of the data window being analyzed and by the sampling rate used when digitizing continuous data, or when collecting discrete data. For a segment of a time series  $f_\lambda$ , of length  $T$  ( $N$  total points), the fundamental period of the segment is assumed to be  $T$ , even though the series is not necessarily periodic, and the lowest frequency represented in spectral analysis of the segment is  $1/T = \nu_1$ .  $1/T$  also represents the highest resolution obtainable in spectral analysis of the time series segment,  $1/T = \nu_1 = \Delta\nu$ .

The sampling interval  $\Delta t$  used in the acquisition of data must be appropriate for the data—two time intervals are needed to define one period. The sampling interval determines the highest obtainable frequency in spectral calculations:  $\nu_N = 1/(2\Delta t) = N/(2T)$ , the Nyquist frequency. Therefore, for a window of a time series  $f_\lambda$  as described above, the frequency limits  $1/T \leq \nu \leq N/(2T)$  exist for spectral analysis of the window.

To avoid aliasing, the contamination of computed spectra by frequencies higher than  $\nu_N$  (see Bendat and Piersol, 1966, pp. 278-280; Waters, 1981, pp. 121-123), sampling must be frequent enough (i.e.,  $\Delta t$  small enough) to have at least two sampling intervals (three sample points) per cycle for the highest frequencies present in a series, not just the highest frequencies of interest. An investigator must, therefore, have some idea of the frequency components that will be present in a series before data are collected. Aliasing problems may also be avoided by appropriate low-pass filtering as part of the data collection process to remove components of frequency higher than the  $\nu_N$  dictated by the chosen sampling interval.



### III. SPECTRAL ANALYSIS OF DISCRETE TIME SERIES

#### A. Fourier Series and Spectral Analysis - Theory

Spectral analysis consists of the description of a given time function in terms of sinusoidal components present in the function. Such analysis is performed using theory developed in part by Joseph Fourier and is referred to as Fourier analysis. According to Fourier theory, a function  $f(t)$ , having a fundamental period of  $2\pi$  and satisfying Dirichlet conditions (see Bâth, 1974, p. 26), can be represented by an infinite series (a Fourier series):

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \quad (1)$$

where  $a_0$ ,  $a_n$ , and  $b_n$  are constants (the Fourier coefficients) which can be represented by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt, \quad (2a)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, \quad (2b)$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt. \quad (2c)$$

In general, for a function with fundamental period  $T$ , the Fourier series and Fourier coefficients are:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}), \quad (3a)$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt, \quad (3b)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi n t}{T} dt, \quad (3c)$$

and

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi n t}{T} dt. \quad (3d)$$

It is the calculation of the Fourier coefficients  $a_n$  and  $b_n$  which is referred to as Fourier analysis. The value  $a_0$  is often referred to as the d.c. (direct current) or steady state component of  $f(t)$ , while  $a_n$  and  $b_n$  are referred to as the a.c. (alternating current) components.

The Fourier transform (a transformation between the time and frequency domains) can be used to calculate the Fourier coefficients of a given time function. The direct Fourier transform is derived in the following (derivation in part from Bâth, 1974, pp. 35-37, and Huang, 1966). The Fourier integral is defined as

$$f(t) = \frac{1}{\pi} \int_0^\infty d\omega \int_{-\infty}^\infty f(\lambda) \cos (\omega(t - \lambda)) d\lambda, \quad (4)$$

where  $\lambda$  is a dummy integration variable. The cosine and sine transforms of a function  $f(\lambda)$  are:

$$a(\omega) = \int_{-\infty}^\infty f(\lambda) \cos \omega \lambda d\lambda \quad (5a)$$

and

$$b(\omega) = \int_{-\infty}^\infty f(\lambda) \sin \omega \lambda d\lambda. \quad (5b)$$

Define a function  $\Phi(\omega)$  where

$$\sin \Phi(\omega) = - \frac{b(\omega)}{\sqrt{a^2(\omega) + b^2(\omega)}}, \quad (6a)$$

from which it follows that

$$\cos \Phi(\omega) = \frac{a(\omega)}{\sqrt{a^2(\omega) + b^2(\omega)}} \quad (6b)$$

and

$$\tan \Phi(\omega) = -\frac{b(\omega)}{a(\omega)}. \quad (6c)$$

Substituting (5a) and (5b) into the Fourier integral (4), with the cosine term expanded, results in

$$f(t) = \frac{1}{\pi} \int_0^{\infty} [a(\omega) \cos \omega t + b(\omega) \sin \omega t] d\omega; \quad (7)$$

see Appendix B for derivation. Introducing equations (6) involving  $\Phi(\omega)$  into equation (7) yields:

$$f(t) = \frac{1}{\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} \cos [\Phi(\omega) + \omega t] d\omega \quad (8)$$

and also

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} e^{i\Phi(\omega)} e^{i\omega t} d\omega \quad (9)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \quad (10)$$

From equations (5), (6), (9), and (10), we can say that

$$F(\omega) = |F(\omega)| e^{i\Phi(\omega)} = [a^2(\omega) + b^2(\omega)]^{1/2} e^{i\Phi(\omega)} \quad (11)$$

$$= [a^2(\omega) + b^2(\omega)]^{1/2} [\cos \Phi(\omega) + i \sin \Phi(\omega)] \quad (12)$$

$$= [a^2(\omega) + b^2(\omega)]^{1/2} \left[ \frac{a(\omega)}{\sqrt{a^2(\omega) + b^2(\omega)}} - i \frac{b(\omega)}{\sqrt{a^2(\omega) + b^2(\omega)}} \right] \quad (13)$$

$$= a(\omega) - i b(\omega) \quad (14)$$

$$= \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - i \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt \quad (15)$$

$$= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt. \quad (16)$$

From the above, the Fourier transform of a time function (direct Fourier transform, Fourier spectrum, complex Fourier transform) is:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \quad (17)$$

where

$$F(\omega) = a(\omega) - i b(\omega) = |F(\omega)| e^{i\Phi(\omega)}. \quad (18)$$

The amplitude spectrum is

$$|F(\omega)| = [a^2(\omega) + b^2(\omega)]^{1/2} \quad (19)$$

and the phase spectrum is

$$\Phi(\omega) = \tan^{-1} \left[ -\frac{b(\omega)}{a(\omega)} \right]. \quad (20)$$

The Fourier spectrum  $F(\omega)$  of a time function is basically an average of  $f(t)e^{-i\omega t}$  over the length of the function, i.e., an average of the components of  $f(t)$  of frequency  $\omega$ .  $F(\omega)$  and  $f(t)$  have the same dimensions.

The real and imaginary parts of  $F(\omega)$ ,  $a(\omega)$  and  $b(\omega)$ , respectively, as shown in (14) and (15), are the cosine and sine transforms of  $f(t)$  as in equations (5). Comparison of these equations to those for the Fourier coefficients,  $a_n$  and  $b_n$  (2), shows that the following relations exist:

$$a(\omega) \rightarrow a_n/2 \quad \text{and} \quad b(\omega) \rightarrow b_n/2. \quad (21)$$

This can be shown as follows:

$$\begin{aligned}
a(\omega) &= \int_{-\infty}^{\infty} f(\lambda) \cos \omega \lambda \, d\lambda = \int_{-\infty}^{\infty} f(t) \cos \frac{2\pi n t}{T} dt \\
&\rightarrow \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi n t}{T} dt = \frac{a_n}{2}.
\end{aligned} \tag{22}$$

The inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \tag{23}$$

This expression can be used to create a time function  $f(t)$  from known spectral components  $F(\omega)$ ; therefore, (23) is referred to as the Fourier synthesis of  $f(t)$ .

#### B. Fourier Series and Spectral Analysis - Discretization

The definitions and derivations given in the previous section are appropriate for continuous functions. The calculation of the Fourier transform  $F(\omega)$  of a given function  $f(t)$  must be approached differently when dealing with a discrete series of observations and when processing data on digital computers. The formulas used must be transformed into discrete form as must data which are recorded in analog form. For data recorded digitally, decisions must be made prior to data acquisition concerning the sampling rate necessary to avoid aliasing problems, as discussed in the previous section. See Båth (1974, Chap. 4) for more details about the equations presented in this section.

In general, a continuous integral may be expressed in discrete form as shown here:

$$\int_0^{N\Delta x} y(x) \, dx \rightarrow \sum_{n=0}^{N-1} y(n\Delta x) \Delta x = \sum_{n=0}^{N-1} y_n \Delta x, \tag{24}$$

where  $y(x)$  is a given curve,  $\Delta x$  is the digitizing interval, and  $N$  is the total number of samples.

An integral with infinite integration limits can be approximated using a form of the trapezoidal rule as follows:

$$\int_{-\infty}^{\infty} y(x) dx \rightarrow \lim_{X \rightarrow \infty} \frac{1}{X} \int_{-X/2}^{X/2} y(x) dx \rightarrow \frac{1}{N\Delta x} \sum_{n=0}^N y(n\Delta x) \Delta x \quad (25)$$

Assuming  $y(0) = y(N)$ , the sum of (25) can be represented as:

$$\frac{1}{N} \sum_{n=1}^N y(n\Delta x) \quad \text{or} \quad \frac{1}{N} \sum_{n=0}^{N-1} y(n\Delta x). \quad (26)$$

Using the general forms stated above, the Fourier coefficients and Fourier series can be expressed in discrete form as:

$$a_0 = \frac{1}{N} \sum_{\lambda=1}^N f_{\lambda}, \quad (27a)$$

$$a_n = \frac{2}{N} \sum_{\lambda=1}^N f_{\lambda} \cos \frac{2\pi n \lambda}{N}, \quad (27b)$$

$$b_n = \frac{2}{N} \sum_{\lambda=1}^N f_{\lambda} \sin \frac{2\pi n \lambda}{N}, \quad (27c)$$

and

$$f_{\lambda} = a_0 + \sum_{n=1}^{N/2} a_n \cos \frac{2\pi n \lambda}{N} + \sum_{n=1}^{N/2} b_n \sin \frac{2\pi n \lambda}{N}. \quad (27d)$$

Applying equation (25) to the direct Fourier transform formula yields:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \rightarrow \frac{1}{N} \sum_{\lambda=1}^N f_{\lambda} e^{-i\omega \lambda}. \quad (28)$$

Equation (28) can be rewritten as follows for ease in computation:

$$F_n = \frac{1}{N} \sum_{\lambda=1}^N f_{\lambda} e^{-i(2\pi n \lambda / N)}. \quad (29)$$

Comparison of (29) to the continuous Fourier transform formulas (14-16) and the Fourier coefficient equations stated above (27) results in:

$$F_n = \frac{1}{2}(a_n - ib_n), \quad (30)$$

i.e., the real and imaginary parts of a Fourier transform are one-half the values of the Fourier coefficients,  $a_n$  and  $b_n$ , respectively. The inverse Fourier transform is also easily represented in discrete form:

$$f_\lambda = \sum_{n=1}^N F_n e^{i(2\pi n\lambda/N)}. \quad (31)$$

### C. Power Spectral Density Function - Definition

The power spectral density is a frequency domain function which is often used to indicate the dominant frequency components present in data (see Bendat and Piersol, 1966, Chap. 3). Before we can define the power spectral density function, we must first introduce the autocorrelation function (Robinson and Treitel, 1980, Chapters 3 and 6). The autocorrelation function for random data gives some indication of the relative amount of power at different frequencies in the data. The autocorrelation function is the expected value of the product of a time series and a time shifted version of that series

$$R(\tau) = E\{f_\lambda f_{\lambda+\tau}\}, \quad (32)$$

which can also be written as

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) f(t+\tau) dt. \quad (33)$$

The autocorrelation function is a real and positive valued even function with the maximum value occurring at  $\tau=0$ . This maximum value corresponds to the mean square value of  $f(t)$ , and represents the energy or power of the time function.

The power spectral density  $S(v)$ , associated with the time function  $f(t)$ , with frequencies defined over the interval  $(-\infty, \infty)$ , is related to the autocorrelation function:

$$R(0) = E\{f^2(t)\} = \int_{-\infty}^{\infty} S(v) dv. \quad (34)$$

The power spectral density function represents how the mean square value of a time function is distributed over the infinite frequency range.  $S(v)$  is called the two-sided power spectral density function of  $f(t)$  because of the frequency interval  $(-\infty, \infty)$  over which it is

defined. It can be shown that the two functions in equation (34) form a Fourier transform pair:

$$R(\tau) = \int_{-\infty}^{\infty} S(\nu) e^{i2\pi\nu\tau} d\nu \leftrightarrow S(\nu) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi\nu\tau} d\tau. \quad (35)$$

Because of the nature of the autocorrelation function and properties of the Fourier transform, the power spectral density function is a real and positive valued even function of frequency. The above transforms may therefore be simplified to:

$$S(\nu) = \int_{-\infty}^{\infty} R(\tau) \cos 2\pi\nu\tau d\tau = 2 \int_0^{\infty} R(\tau) \cos 2\pi\nu\tau d\tau \quad (36)$$

and

$$R(\tau) = 2 \int_0^{\infty} S(\nu) \cos 2\pi\nu\tau d\nu. \quad (37)$$

Assuming that  $R(\tau)$  exists, and that it has the Fourier transform  $S(\nu)$ , the power spectral density function can be defined as the direct Fourier transform of the autocorrelation function:

$$S(\nu) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi\nu\tau} d\tau. \quad (38)$$

The physically realizable one-sided power spectral density function  $G(\nu)$  for the frequency interval  $(0, \infty)$  is defined as:

$$G(\nu) = 2S(\nu), \quad 0 \leq \nu < \infty. \quad (39)$$

The correspondence between the one-sided power spectral density function and the autocorrelation function is:

$$G(\nu) = 4 \int_0^{\infty} R(\tau) \cos 2\pi\nu\tau d\tau \quad (40)$$

and



$$R(\tau) = \int_0^{\infty} G(v) \cos 2\pi v \tau \, dv. \quad (41)$$

#### D. Power Spectral Density Function - Estimation

To get the power spectral density function into a form that can be computed directly from observational data, without calculation of the autocorrelation function, we must consider the estimation of sample parameters (Bendat and Piersol, 1966, Chap. 5). The estimation of a previously defined parameter will be indicated by a hat (^) symbol over the parameter symbol. For example,  $\hat{e}$  is an estimate of  $e$ , where  $e$  is any parameter, such as the mean value or power spectral density of  $f_\lambda$ , a sample time series existing for the finite time interval  $T$ . Estimators are often defined arbitrarily and may not clearly give a correct estimation of a given parameter.

Consider the time function  $f(t)$ ; the autocorrelation function  $R(\tau)$  can be estimated by the sample autocorrelation function  $\hat{R}(\tau)$ , for  $f(t)$  existing over the time interval  $T$  by

$$\hat{R}(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} f(t) f(t+\tau) \, dt, \quad 0 \leq \tau < T. \quad (42)$$

Assuming that the data exist for time  $T+\tau$ , the sample autocorrelation function can be estimated by

$$\hat{R}(\tau) = \frac{1}{T} \int_0^T f(t) f(t+\tau) \, dt, \quad 0 \leq \tau < T. \quad (43)$$

The formula for the estimation of the power spectral density function is not defined directly in terms of the autocorrelation function, but takes into account the fact that the autocorrelation function is related to the mean square value of a time function. For a time function  $f(t)$  with zero mean, existing over a time interval  $T$ , the estimate of the one-sided power spectral density function  $\hat{G}(v)$  describes the time average of  $f^2(t)$  in terms of its frequency components, in the frequency interval  $(v-(B_e/2), v+(B_e/2))$ , where  $B_e$  is the frequency bandwidth.

The mean square value of  $f(t)$  within the bandwidth  $B_e$  centered at  $v$  is estimated by

$$\hat{\Psi}^2(v, B_e) = \frac{1}{T} \int_0^T f^2(t, v, B_e) \, dt. \quad (44)$$

The power spectral density is defined as

$$\begin{aligned}
 G(v) &= \lim_{B_e \rightarrow 0} \frac{\Psi^2(v, B_e)}{B_e} = \lim_{\substack{T \rightarrow \infty \\ B_e \rightarrow 0}} \frac{1}{B_e T} \int_0^T f^2(t, v, B_e) dt \\
 &= \lim_{\substack{T \rightarrow \infty \\ B_e \rightarrow 0}} \hat{G}(v).
 \end{aligned} \tag{45}$$

From equations (44) and (45), the sample estimate for the one-sided power spectral density function  $\hat{G}(v)$  is:

$$\hat{G}(v) = \frac{\hat{\Psi}^2(v, B_e)}{B_e} = \frac{1}{B_e T} \int_0^T f^2(t, v, B_e) dt. \tag{46}$$

The limits in equation (45) must be evaluated to obtain the true function  $G(v)$ ;  $\hat{G}(v)$  is generally a biased estimate of  $G(v)$ . The power spectral density function estimate  $\hat{G}(v)$  is one-sided and is related to the mathematical two-sided power spectral density function estimate defined for positive and negative frequencies as shown here:

$$\hat{S}(v) = \hat{S}(-v) = \frac{\hat{G}(v)}{2}. \tag{47}$$

The above formula (46) for the estimate of the power spectral density function can be discretized as follows, taking into account that the bandwidth  $B_e$  is equivalent to the Nyquist frequency  $v_N$ :

$$\hat{G}(v) = \frac{1}{B_e T} \int_0^T f^2(t, v, B_e) dt \tag{48a}$$

$$\rightarrow \frac{1}{B_e T} \sum_{\lambda=0}^{N-1} f_{\lambda}^2 \Delta \lambda = \frac{1}{B_e T} \sum_{\lambda=0}^{N-1} f_{\lambda}^2 \frac{T}{N} \tag{48b}$$

$$= \frac{1}{v_N N} \sum_{\lambda=0}^{N-1} f_{\lambda}^2 = \frac{1}{(N/2T)N} \sum_{\lambda=0}^{N-1} f_{\lambda}^2 \tag{48c}$$

$$= \frac{2T}{N^2} \sum_{\lambda=0}^{N-1} f_{\lambda}^2. \quad (49)$$

The estimate of the one-sided power spectral density function as defined above is related to the Fourier spectrum of a time series. Working from the formula for the discrete Fourier transform of  $f_{\lambda}$ ,  $F_n = (a_n - i b_n)/2$ , where  $a_n$  and  $b_n$  are the Fourier coefficients of  $f_{\lambda}$ , the relation can be shown as follows:

$$F_n = \frac{1}{2} (a_n - i b_n) \quad (50)$$

$$F_n^2 = \frac{1}{4} (a_n - i b_n)(a_n + i b_n) = \frac{1}{4} (a_n^2 + b_n^2) \quad (51a)$$

$$= \frac{1}{4} \left[ \left( \frac{2}{N} \sum_{\lambda=1}^N f_{\lambda} \cos \frac{2\pi n \lambda}{N} \right)^2 + \left( \frac{2}{N} \sum_{\lambda=1}^N f_{\lambda} \sin \frac{2\pi n \lambda}{N} \right)^2 \right] \quad (51b)$$

$$= \frac{1}{4} \left( \frac{4}{N^2} \sum_{\lambda=1}^N f_{\lambda} \cos \frac{2\pi n \lambda}{N} \sum_{\lambda=1}^N f_{\lambda} \cos \frac{2\pi n \lambda}{N} \right) \\ + \frac{1}{4} \left( \frac{4}{N^2} \sum_{\lambda=1}^N f_{\lambda} \sin \frac{2\pi n \lambda}{N} \sum_{\lambda=1}^N f_{\lambda} \sin \frac{2\pi n \lambda}{N} \right) \quad (51c)$$

$$= \frac{1}{N^2} \left[ \sum_{\lambda=1}^N \sum_{\lambda=1}^N f_{\lambda}^2 \cos^2 \left( \frac{2\pi n \lambda}{N} \right) + \sum_{\lambda=1}^N \sum_{\lambda=1}^N f_{\lambda}^2 \sin^2 \left( \frac{2\pi n \lambda}{N} \right) \right] \quad (51d)$$

$$= \frac{1}{N^2} \left[ \sum_{\lambda=1}^N \sum_{\lambda=1}^N f_{\lambda}^2 \left( \cos^2 \left( \frac{2\pi n \lambda}{N} \right) + \sin^2 \left( \frac{2\pi n \lambda}{N} \right) \right) \right] \quad (51e)$$

$$= \frac{1}{N^2} \sum_{\lambda=1}^N f_{\lambda}^2. \quad (52)$$

Comparing equation (52) with the discrete equation for the estimate of the power spectral density (49) yields the following relation:

$$\widehat{G}_n = 2TF_n^2. \quad (53)$$

This can be rewritten, to ease comparison to the Fourier transform of a time series of interest:

$$\text{PSD}_n = \widehat{G}_n = \frac{2T}{N^2} \left| \sum_{\lambda=1}^N f_{\lambda} e^{-i2\pi n \lambda / N} \right|^2. \quad (54)$$

The units of the power spectral density function are (units of Fourier transform)<sup>2</sup>/Hz.

#### E. Application to Residual Acceleration Data

Residual acceleration data have been collected in orbiting space laboratories with a variety of instruments. The data are typically recorded and distributed to investigators in discrete form. Sampling rates vary considerably from one experiment to the next, depending on the specific goal of the experiment. Typical sampling rates range from 12.5 to 500 samples per second. Sampling is often done at higher rates than required to obtain a particular maximum frequency. For example, if the highest frequency of interest to an investigator is 100 Hz, data may be collected at 500 samples per second and then lowpass filtered down to 100 Hz. Such a procedure is useful when investigators are not sure of the maximum frequencies that compose the process being sampled. This also allows for higher resolution of frequency domain data than would be available if a sampling rate of only 200 Hz was used.

Before any specific Fourier analysis can be done on accelerometer data, some pre-processing is usually required. Pre-processing is performed so that the output represents as closely as possible the actual residual accelerations experienced at the recording site. Artificial signals, or noise, can be introduced from a multitude of sources. Manufacturers of recording devices typically furnish users with specific corrections for deviations from pure signal caused by temperature variations, instrumental bias, and other factors. Filtering can also be applied to data to minimize the effects of aliasing, instrumental noise, and other known noise sources. Specific filtering techniques will not be discussed here, but discussions of filtering can be found in most signal processing texts, see B  th (1974), Bendat and Piersol (1980), and Cadzow (1987).

Once such pre-processing has been applied to residual acceleration data, Fourier analysis can be applied in order to determine what frequency components are significantly present. Windows must be chosen of a length appropriate to what minimum frequencies one wants to see, as discussed in the Sampling Considerations section. Many Fourier transform algorithms are most efficient when the number of samples is highly composite, e.g., a power of two, which places further restrictions on the window length. After pre-processing, the data are assumed free of artificial signals. The results of inadequate correction for instrumental bias can often be seen in Fourier transformed data. An artificial bias from the zero point will cause a significant steady state component. Other incorrect processing, however, is not as easy to identify, but may be manifested as a simple bias, or as long-term variations or trends.

Most standard Fourier transform computer programs use algorithms based on the discrete Fourier transform formula

$$F_n = \frac{1}{N} \sum_{\lambda=1}^N f_{\lambda} e^{-i(2\pi n\lambda/N)}, \quad (29)$$

and may give as output the cosine and sine transforms,  $a(\omega)$  and  $b(\omega)$ , respectively, or the amplitude and phase spectra:

$$F_n = \sqrt{a^2(\omega) + b^2(\omega)} \quad \Phi_n = \tan^{-1} \left[ -\frac{b(\omega)}{a(\omega)} \right]. \quad (55)$$

The Fourier coefficients  $a_n$  and  $b_n$  which differ from the cosine and sine transforms by a factor of 2 may also be output, so it is important to completely understand the transform routine being used. See, for example, Bloomfield (1976, Chap. 4), Elliott and Rao (1982), and Cadzow (1987, Chap. 4) for information on Fourier transform algorithms.

Given the Fourier transform  $F_n$ , the original series may be recreated using the discrete inverse Fourier transform (31) or it may be synthesized using the relation between  $F_n$  and the Fourier coefficients (30) and the discrete Fourier series equation (27d). Using equation (54), output from Fourier transform programs can be used to form the power spectral density function of the time series considered which can be investigated in addition to, or as an alternative to, the Fourier spectrum.

## IV. TRANSFORMATION OF COORDINATE AXES

### A. Transformation of Coordinate Axes - Theory

Observational time series data are usually recorded in a stationary set of coordinate axes, which may be located and oriented arbitrarily. Such data can be analyzed in terms of alternative axes by a relatively simple transformation operation, as long as the transformation parameters between the axes of interest and the recording axes are known.

One common form of coordinate axes transformation is that which transforms axes that share a common origin and differ by an angle of rotation  $\theta$  about a common axis; see Figure 1. In the transformation of axes, a known vector  $\mathbf{g}$ , such as a residual acceleration measurement, is written in terms of some (primed) coordinate axes which are related to the original (unprimed) coordinate axes by a known rotation angle. That is,  $\mathbf{g}$  has some coordinates in the unprimed axes ( $g_1, g_2, g_3$ ), and some other coordinates in the new, primed axes ( $g_1', g_2', g_3'$ ).

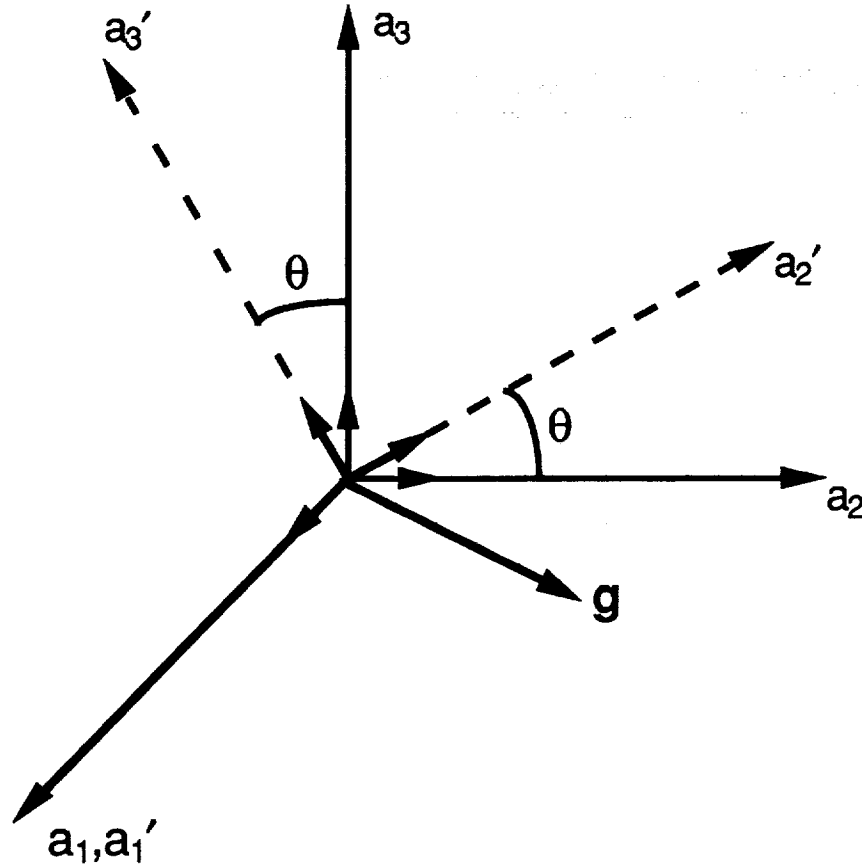


Figure 1. Axes  $a_2'$  and  $a_3'$  are rotated by an angle of  $\theta$  about the  $a_1$  axis.  $\mathbf{g}$  is some vector with coordinates  $(g_1, g_2, g_3)$  in the unprimed axes and  $(g_1', g_2', g_3')$  in the primed axes. Unit vectors  $\mathbf{e}$  and  $\mathbf{e}'$  are marked by short vectors along the coordinate axes.

Referring to the vector  $\mathbf{g}$  in terms of some new coordinate axes is equivalent to a change in basis of a vector space from an original basis to a new basis (Anton, 1981, Chap. 4). The vector  $\mathbf{g}$  is rewritten as  $\mathbf{g}'$  in terms of the new coordinate axes.  $\mathbf{g}'$  is obtained by the application of a transition matrix to  $\mathbf{g}$ :

$$\mathbf{g}' = \mathbf{R}\mathbf{g}. \quad (56)$$

$\mathbf{R}$  is called the transformation matrix (or rotation matrix). The columns of  $\mathbf{R}$  are the components of the old basis vectors relative to the new basis. Consider, for example, the unit vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $\mathbf{e}_1'$ , and  $\mathbf{e}_2'$  of two coordinate axes rotated with respect to one another by an angle  $\theta$ . The transition matrix can be constructed by writing  $\mathbf{e}_1$  and  $\mathbf{e}_2$  in terms of the primed axes:

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{e}_1' \cos \theta - \mathbf{e}_2' \sin \theta \\ \mathbf{e}_2 &= \mathbf{e}_1' \sin \theta + \mathbf{e}_2' \cos \theta \end{aligned} \quad (57)$$

which yields the transition matrix:

$$\mathbf{R} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \quad (58)$$

This is the transition matrix from unprimed to primed coordinate axes:

$$\mathbf{g}' = \mathbf{R}\mathbf{g}. \quad (59)$$

This can be checked by writing out the elements of  $\mathbf{g}'$ :

$$\begin{aligned} g_1' &= g_1 \cos \theta + g_2 \sin \theta \\ g_2' &= -g_1 \sin \theta + g_2 \cos \theta \end{aligned} \quad (60)$$

and comparing them to the equations for  $\mathbf{e}_1'$  and  $\mathbf{e}_2'$  in terms of the unprimed axes:

$$\begin{aligned} \mathbf{e}_1' &= \mathbf{e}_1 \cos \theta + \mathbf{e}_2 \sin \theta \\ \mathbf{e}_2' &= -\mathbf{e}_1 \sin \theta + \mathbf{e}_2 \cos \theta. \end{aligned} \quad (61)$$

Equations (60) and (61) are equivalent. Equations (61) could be used to create the transition matrix for primed to unprimed coordinate axes, which would be equal to the inverse ( $\mathbf{R}^{-1} = \mathbf{R}^t$ ) of the transition matrix from unprimed to primed coordinate axes. The process is easily extended to three dimensions (Anton, 1981, Chap. 4).

In general, for a coordinate system defined by three mutually perpendicular axes, the transition matrix can be constructed from the direction cosines of the primed axes with respect to the unprimed axes:

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = R_{ij} \quad (62)$$

where the first subscript denotes a primed axis and the second an unprimed axis (Frederick and Chang, 1972, Chap. 1). For example,  $R_{12}$  is the cosine of the angle between the  $a_1'$ -axis and the  $a_2$ -axis. Hence, the transition of a vector from an unprimed system to a primed system can be obtained by

$$g_i' = R_{ij} g_j. \quad (63)$$

The use of equation (62) is preferred over that of equation (58) because (62) is more general and can be used for cases where the rotation is around any line through the origin, not only around a shared axis.

#### B. Application to Residual Acceleration Data

The transformation of coordinate axes can be a very useful operation in the analysis of data. In the case of the analysis of residual acceleration data in conjunction with experimental data, the ability to consider data in terms of different coordinate axes is a necessity. Experiments run in orbiting space laboratories, including accelerometer systems, are often oriented in a manner convenient to space restrictions. This results in a plethora of experimental coordinate systems for one mission.

While the total acceleration vector has the same magnitude in any coordinate system, the disturbance level in some particular direction may be of interest to investigators. Some experiments may be more sensitive to disturbances in one direction than in another direction. In the analysis of the results of such experiments, it is beneficial to transform the accelerometer data into a set of axes coincident with the direction of interest. The magnitude of disturbances in the direction of interest can then be obtained. If the sources of disruptive disturbances can be identified, and if these disturbances tend to be uni-directional, such knowledge could be used in the future when positioning experiments in orbiters.



## V. CONCLUSIONS

Various aspects of residual acceleration data are of interest to investigators. The mean, variance, and minimum and maximum values can be obtained from the data as collected in the time domain. Additional information can be obtained by looking at data from a different perspective. Dominant frequency components can be identified following a transformation of data into the frequency domain (amplitude spectrum or power spectral density) using Fourier transform methods. Information obtained in both the time and frequency domains can be used by investigators to determine what magnitude disturbances and what frequency modes are disruptive to their experiments.

The orientation of accelerometer recording axes and of recorded accelerations is important in the analysis of low-gravity experiments. The ability to refer to acceleration data in terms of different coordinate axes (such as those of a separate experiment) is useful in the post-flight analysis of experiments.

Methods such as those discussed here can be implemented in the analysis of residual acceleration data collected in orbiting space laboratories and used to support analysis of experiments run under low-gravity conditions.

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## APPENDIX A - NOMENCLATURE

$a_n, b_n$	Fourier coefficients
$a(\omega)$	cosine transform of time function
$b(\omega)$	sine transform of time function
$\mathbf{e}_1, \mathbf{e}_2$	unit vectors
$f(t)$	function of time, $t$
$f_\lambda$	time series with increment $\lambda$ , digitized version of $f(t)$
$F(\omega)$	Fourier transform of $f(t)$
$F_n$	Fourier transform of $f_\lambda$
$\mathbf{g}$	vector with coordinates $(g_1, g_2, g_3)$
$G(v)$	one-sided power spectral density function
$i$	$\sqrt{-1}$
$N$	length of time series in number of samples
$R_{ij}$	transformation matrix = $\mathbf{R}$
$R(\tau)$	autocorrelation function of a time series
$S(v)$	two-sided power spectral density function
$T$	length of time series or function in seconds, also fundamental period of time series or function
$\Delta t$	sampling interval
$\Phi(\omega)$	phase spectrum of $f(t)$
$\theta$	angle of rotation between coordinate axes
$\tau$	time shift
$\nu$	cyclic frequency
$\nu_N$	Nyquist frequency
$\omega$	angular frequency, $\omega=2\pi\nu$

^ over a symbol in the text represents an estimate of the parameter.

## APPENDIX B - DETAILED DERIVATIONS

The derivation of equations (9) and (10) in the main text from the Fourier integral (4) can be done as follows. Given the Fourier integral

$$f(t) = \frac{1}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(\lambda) \cos(\omega(t - \lambda)) d\lambda, \quad (\text{B1})$$

the cosine term can be expanded, yielding:

$$f(t) = \frac{1}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(\lambda) [\cos \omega t \cos \omega \lambda + \sin \omega t \sin \omega \lambda] d\lambda \quad (\text{B2})$$

$$= \frac{1}{\pi} \int_0^{\infty} d\omega \left[ \int_{-\infty}^{\infty} [f(\lambda) \cos \omega t \cos \omega \lambda + f(\lambda) \sin \omega t \sin \omega \lambda] d\lambda \right] \quad (\text{B3})$$

$$= \frac{1}{\pi} \int_0^{\infty} d\omega \left[ \int_{-\infty}^{\infty} f(\lambda) \cos \omega \lambda \cos \omega t d\lambda + \int_{-\infty}^{\infty} f(\lambda) \sin \omega \lambda \sin \omega t d\lambda \right] \quad (\text{B4})$$

$$= \frac{1}{\pi} \int_0^{\infty} [a(\omega) \cos \omega t + b(\omega) \sin \omega t] d\omega. \quad (\text{B5})$$

Equation (B5) is the same as equation (7) in the main text. Introducing equations (6) from the main text for  $\Phi(\omega)$ ,  $a(\omega)$ , and  $b(\omega)$  yields:

$$f(t) = \frac{1}{\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} \cos \Phi(\omega) \cos \omega t - [a^2(\omega) + b^2(\omega)]^{1/2} \sin \Phi(\omega) \sin \omega t d\omega \quad (\text{B6})$$

$$= \frac{1}{\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [\cos \Phi(\omega) \cos \omega t - \sin \Phi(\omega) \sin \omega t] d\omega \quad (\text{B7})$$

$$= \frac{1}{\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [\cos(\Phi(\omega) + \omega t)] d\omega. \quad (\text{B8})$$

Equation (B8), which is equation (8) in the main text, can be rewritten as follows:

$$f(t) = \frac{1}{2\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i(\Phi(\omega) + \omega t)} + e^{-i(\Phi(\omega) + \omega t)}] d\omega \quad (\text{B9})$$

$$= \frac{1}{2\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i(\Phi(\omega) + \omega t)}] d\omega \\ + \frac{1}{2\pi} \int_{-\infty}^0 [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i(\Phi(\omega) + \omega t)}] d\omega, \quad (\text{B10})$$

by reversal of the limits of integration. Manipulation of the exponential terms yields:

$$f(t) = \frac{1}{2\pi} \int_0^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i\omega t} e^{i\Phi(\omega)}] d\omega \\ + \frac{1}{2\pi} \int_{-\infty}^0 [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i\omega t} e^{i\Phi(\omega)}] d\omega. \quad (\text{B11})$$

Addition of the two integrals results in equations (9) and (10) of the main text, the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [a^2(\omega) + b^2(\omega)]^{1/2} [e^{i\omega t} e^{i\Phi(\omega)}] d\omega \quad (\text{B12})$$

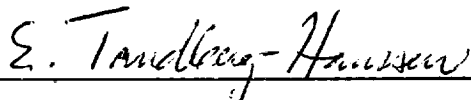
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \quad (\text{B13})$$

## APPROVAL

### ANALYSIS TECHNIQUES FOR RESIDUAL ACCELERATION DATA

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The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.



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